Chapter 4

Representing Functions

4.1 Robinson Arithmetic

We start by describing a first order theory called *Robinson Arithmetic*.

Definition 1.1: Robinson Arithmetic

The signature of the language is $\mathcal{L}_{\mathcal{A}} = \{0, S, +, \cdot\}$ where

- 0 is a constant symbol,
- S is a unary function symbol[a],
- $+$ and $\cdot$ are binary function symbols[b].

The axioms are

axiom 1. $\forall x\ Sx \neq 0$

axiom 2. $\forall x\ \exists y\ (x \neq 0 \rightarrow Sy = x)$

axiom 3. $\forall x\ \forall y\ (Sx = Sy \rightarrow x = y)$

axiom 4. $\forall x\ x{+}0 = x$

axiom 5. $\forall x\ \forall y\ \Big(x{+}Sy = S(x{+}y)\Big)$

axiom 6. $\forall x\ x{\cdot}0 = 0$

axiom 7. $\forall x\ \forall y\ \Big(x{\cdot}Sy = (x{\cdot}y){+}x\Big)$

[a] for any terms of $\mathcal{L}_{\mathcal{A}}$ t, we use the notation St instead of $S(t)$.

[b] for any terms of $\mathcal{L}_{\mathcal{A}}$ t_0, t_1, we use the notation $t_0{+}t_1$ (respectively $t_0{\cdot}t_1$) instead of $+(t_0, t_1)$ (respectively $\cdot(t_0, t_1)$).

Example 1.1

The *standard model* of Robinson Arithmetic is

$$\langle \mathbb{N}, 0, S, +, \cdot \rangle$$

where S is the successor function, $+$ is the usual addition, and $\cdot$ is the customary multiplication.
By abuse of notation we identify $\mathbb{N}$ with the model $\langle \mathbb{N}, 0, S, +, \cdot \rangle$. So that, for instance, we will write

$$\mathbb{N} \models \varphi$$

instead of the correct notation

$$\langle \mathbb{N}, 0, S, +, \cdot \rangle \models \varphi.$$

Example 1.2

A very simple non standard model of Robinson arithmetic $\mathcal{M}$ in which

- $S^{\mathcal{M}}$ admits a fixed point.
- $\cdot^{\mathcal{M}}$ is not commutative.

$$\mathcal{M} = \langle \mathbb{N} \cup \{a\}, 0^{\mathcal{M}}, S^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}} \rangle$$

Where $0^{\mathcal{M}} = 0$ and the operations $S^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}$ are defined the usual way on the integers. i.e

- $S^{\mathcal{M}} \restriction \mathbb{N} = S$
- $+^{\mathcal{M}} \restriction \mathbb{N} \times \mathbb{N} = +$
- $\cdot^{\mathcal{M}} \restriction \mathbb{N} \times \mathbb{N} = \cdot$

And when the unique non standard integer a is involved:

- $S^{\mathcal{M}} a = a$
- $\alpha \cdot^{\mathcal{M}} a = a$ (any $\alpha \in \mathbb{N} \cup \{a\}$)
- $a +^{\mathcal{M}} \alpha = \alpha +^{\mathcal{M}} a = a$ (any $\alpha \in \mathbb{N} \cup \{a\}$)
- $a \cdot^{\mathcal{M}} 0 = 0$
- $a \cdot^{\mathcal{M}} \alpha = a$ (any $\alpha \in \big(\mathbb{N} \smallsetminus \{0\}\big) \cup \{a\}$)

We verify that

$$\mathcal{M} \models \mathcal{R}ob.$$

axiom 1. since the only non standard integer verifies $S^{\mathcal{M}}a = a$ we have

$$\mathcal{M} \models \forall x\ Sx \neq 0$$

axiom 2. since every standard integer different from 0 has a predecessor, and $S^{\mathcal{M}}a = a$, we have

$$\mathcal{M} \models \forall x\ \exists y\ (x \neq 0 \to Sy = x)$$

axiom 3. holds for standard integers, and for every $n \in \mathbb{N}$ we have $S^{\mathcal{M}}a \neq S^{\mathcal{M}}n$, thus

$$\mathcal{M} \models \forall x\ \forall y\ (Sx = Sy \to x = y)$$

axiom 4. holds for standard integers, and we have $a +^{\mathcal{M}} 0 = a$, hence

$$\mathcal{M} \models \forall x\ x + 0 = x$$

axiom 5. if $k, n \in \mathbb{N}$, then $k +^{\mathcal{M}} S^{\mathcal{M}}n = S^{\mathcal{M}}(k +^{\mathcal{M}} n)$ holds. If $\alpha \in \mathbb{N} \cup \{a\}$, we have

- $a +^{\mathcal{M}} S^{\mathcal{M}}\alpha = a$
- $S^{\mathcal{M}}(a +^{\mathcal{M}} \alpha) = S^{\mathcal{M}}a = a$
- $\alpha +^{\mathcal{M}} S^{\mathcal{M}}a = \alpha +^{\mathcal{M}} a = a$
- $S^{\mathcal{M}}(\alpha +^{\mathcal{M}} a) = S^{\mathcal{M}}a = a$

therefore we have

$$\mathcal{M} \models \forall x\ \forall y\ \Big(x + Sy = S(x + y)\Big).$$

axiom 6. if $\alpha \in \mathbb{N} \cup \{a\}$, we have $\alpha \cdot^{\mathcal{M}} 0 = 0$. Thus

$$\mathcal{M} \models \forall x\ x \cdot 0 = 0$$

axiom 7. if $k, n \in \mathbb{N}$ and $\alpha \in \mathbb{N} \cup \{a\}$, then

- $a \cdot^{\mathcal{M}} S^{\mathcal{M}}\alpha = a$
- $\alpha \cdot^{\mathcal{M}} S^{\mathcal{M}}a = a$
- $(a \cdot^{\mathcal{M}} \alpha) +^{\mathcal{M}} a = a$
- $(\alpha \cdot^{\mathcal{M}} a) +^{\mathcal{M}} \alpha = a$
- $k \cdot^{\mathcal{M}} S^{\mathcal{M}}n = (k \cdot^{\mathcal{M}} n) +^{\mathcal{M}} k$

so we have

$$\mathcal{M} \models \forall x\ \forall y\ \Big(x \cdot Sy = (x \cdot y) + x\Big).$$

Notation 1.1

For any integer n, we write $\mathbf{n}$ for the $\mathcal{L}_{\mathcal{A}}$-term $\underbrace{S\ldots S}_{n}0$.

Example 1.3

Let us show that the following holds for all integers k:

$$\mathcal{R}ob. \vdash_c \forall x \left(0+x = \mathbf{k} \longrightarrow x = \mathbf{k}\right) \tag{4.1}$$

We will make use of the excluded middle:

$$\vdash_c \forall x \quad (x = 0 \ \vee \ x \neq 0)$$

and distinguish between the two cases.
The proof is by induction on k:

if $\mathbf{k = 0}$:

if $\mathbf{x = 0}$: since $\vdash_c 0 = \mathbf{0}$ holds, we obtain

$$\mathcal{R}ob. \vdash_c 0+0 = \mathbf{0} \longrightarrow 0 = \mathbf{0}$$

if $\mathbf{x \neq 0}$: by $\boxed{2 \quad \forall x\, \exists y\, (x \neq 0 \to Sy = x)}$ it is enough to show

$$\mathcal{R}ob. \vdash_c \forall y \left(0+Sy = \mathbf{0} \longrightarrow Sy = \mathbf{0}\right)$$

by $\boxed{5 \quad \forall x\, \forall y\, \left(x+Sy = S(x+y)\right)}$ this comes down to establishing

$$\mathcal{R}ob. \vdash_c \forall y \left(S(0+y) = \mathbf{0} \longrightarrow Sy = \mathbf{0}\right)$$

which is immediate by $\boxed{1 \quad \forall x\, Sx \neq 0}$.

if $\mathbf{k = n+1}$:

if $\mathbf{x = 0}$: we need to show

$$\mathcal{R}ob. \vdash_c 0+0 = \mathbf{n+1} \longrightarrow 0 = \mathbf{n+1}.$$

By $\boxed{④ \quad \forall x\ x{+}0 = x}$ this comes down to showing

$$\mathcal{R}ob.\ \vdash_c\ 0 = n+1\ \longrightarrow\ 0 = n+1;$$

which obviously holds since the following already holds:

$$\vdash_c\ 0 = n+1\ \longrightarrow\ 0 = n+1.$$

if x $\neq$ 0: by $\boxed{② \quad \forall x\ \exists y\ (x \neq 0 \to Sy = x)}$ it is enough to show

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}Sy = n+1\ \longrightarrow\ Sy = n+1\big)$$

by $\boxed{⑤ \quad \forall x\ \forall y\ \big(x{+}Sy = S(x{+}y)\big)}$ this comes down to establishing

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(S(0{+}y) = n+1\ \longrightarrow\ Sy = n+1\big)$$

which is exactly

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(S(0{+}y) = Sn\ \longrightarrow\ Sy = Sn\big)$$

by $\boxed{③ \quad \forall x\ \forall y\ (Sx = Sy \to x = y)}$ this amounts to showing

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}y = n\ \longrightarrow\ Sy = Sn\big).$$

The induction hypothesis gives

$$\mathcal{R}ob.\ \vdash_c\ \forall y\ \big(0{+}y = n\ \longrightarrow\ y = n\big)$$

from where we immediately get what we want.

Example 1.4

Let us show that the following holds for all integers k:

$$\mathcal{R}ob.\ \vdash_c\ \forall x \forall y\ \big(Sy{+}x = k\ \longrightarrow\ S(y{+}x) = k\big). \tag{4.2}$$

We make use of the excluded middle:

$$\vdash_c\ \forall x\ \ \big(x = 0\ \vee\ x \neq 0\big)$$

and distinguish between the two cases.

if x = 0: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \left(Sy+0 = k \longrightarrow S(y+0) = k\right)$$

which is immediate by $\boxed{(4) \quad \forall x\ x+0 = x}$.

if x ≠ 0: it is enough to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy+Sz = k \longrightarrow S(y+Sz) = k\right).$$

The proof goes by induction on k:

if k = 0: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy+Sz = 0 \longrightarrow S(y+Sz) = 0\right).$$

By $\boxed{(5) \quad \forall x\ \forall y\ \left(x+Sy = S(x+y)\right)}$ this comes down to

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(S(Sy+z) = 0 \longrightarrow S(y+Sz) = 0\right)$$

which trivially holds by $\boxed{(1) \quad \forall x\ Sx \neq 0}$.

if k = n + 1: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy+Sz = n+1 \longrightarrow S(y+Sz) = n+1\right).$$

By $\boxed{(5) \quad \forall x\ \forall y\ \left(x+Sy = S(x+y)\right)}$ this comes down to

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(S(Sy+z) = Sn \longrightarrow S(y+Sz) = n+1\right).$$

By $\boxed{(3) \quad \forall x\ \forall y\ (Sx = Sy \rightarrow x = y)}$ this amounts to proving

$$\mathcal{R}ob. \vdash_c \forall y \forall z \left(Sy+z = n \longrightarrow S(y+Sz) = n+1\right).$$

if z = 0: we need to show

$$\mathcal{R}ob. \vdash_c \forall y \left(Sy+0 = n \longrightarrow S(y+S0) = n+1\right).$$

By $\boxed{(4) \quad \forall x\ x+0 = x}$ and $\boxed{(5) \quad \forall x\ \forall y\ \left(x+Sy = S(x+y)\right)}$ this comes down to showing

$$\mathcal{R}ob. \vdash_c \forall y \left(Sy = n \longrightarrow SSy = n+1\right).$$

which holds by definition.

if z $\neq$ 0: what we need to prove is equivalent to

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow S(y+SSz') = n+1\right).$$

By $\boxed{③ \quad \forall x\, \forall y\, (Sx = Sy \rightarrow x = y)}$ this comes down to showing

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow y+SSz' = n\right).$$

The induction hypothesis yields

$$\mathcal{R}ob. \vdash_c \forall y \forall z' \left(Sy+Sz' = n \longrightarrow S(y+Sz') = n\right).$$

from where we easily get the result by $\boxed{⑤ \quad \forall x\, \forall y\, \big(x+Sy = S(x+y)\big)}$.

4.2 Representable Functions

Definition 2.1

Let $f \in \mathbb{N}^{(\mathbb{N}^n)}$ and $\varphi(x_0, x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathcal{A}}$-formula whose free variables are among $\{x_0, x_1, \ldots, x_n\}$.
$\varphi(x_0, x_1, \ldots, x_n)$ represents the function f if for all $i_1, \ldots, i_n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f(i_1, \ldots, i_n) = x_0 \longleftrightarrow \varphi(x_0, i_1, \ldots, i_n)\Big).$$

Definition 2.2

Let $A \subseteq \mathbb{N}^n$ and $\varphi(x_1, \ldots, x_n)$ be any $\mathcal{L}_{\mathcal{A}}$-formula whose free variables are among $\{x_1, \ldots, x_n\}$.
$\varphi(x_1, \ldots, x_n)$ represents the set A if for all $i_1, \ldots, i_n \in \mathbb{N}$ we have:

- if $(i_1, \ldots, i_n) \in A$, then $\mathcal{R}ob. \vdash_c \varphi(i_1, \ldots, i_n)$;
- if $(i_1, \ldots, i_n) \notin A$, then $\mathcal{R}ob. \vdash_c \neg\varphi(i_1, \ldots, i_n)$.

Proposition 2.1

For any $A \subseteq \mathbb{N}^n$,

A is representable if and only if χ_A is representable.

Proof of Proposition 2.1:

($\Rightarrow$) If A is represented by $\varphi(x_1, \ldots, x_n)$, then χ_A is represented by

$$\big(x_0 = 1 \wedge \varphi(x_1, \ldots, x_n)\big) \vee \big(x_0 = 0 \wedge \neg\varphi(x_1, \ldots, x_n)\big).$$

($\Leftarrow$) If χ_A is represented by $\varphi(x_0, x_1, \ldots, x_n)$, then A is represented by

$$\varphi(S0, x_1, \ldots, x_n).$$

$\square$

Example 2.1

The constant function $f \in \mathbb{N}^{(\mathbb{N}^n)}$ defined by $f(i_1, \ldots, i_n) = k$ (any $i_1, \ldots, i_n \in \mathbb{N}$) is represented by the following formula of the form $\varphi(x_0, i_1, \ldots, i_n)$:

$$x_0 = k.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f(i_1, \ldots, i_n) = x_0 \longleftrightarrow x_0 = k\Big).$$

which is exactly

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(k = x_0 \longleftrightarrow x_0 = k\Big).$$

Example 2.2

The projection $\pi_j^n \in \mathbb{N}^{(\mathbb{N}^n)}$is represented by the formula:

$$x_0 = i_j.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(\pi_j^n(i_1, \ldots, i_n) = x_0 \longleftrightarrow x_0 = i_j\Big).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_j = x_0 \longleftrightarrow x_0 = i_j \right).$$

Example 2.3

The successor $S \in \mathbb{N}^{\mathbb{N}}$ is represented by the formula:

$$x_0 = Sx_1.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S(i) = x_0 \longleftrightarrow x_0 = Si \right).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S\,\overbrace{S \dots S}^{i}\,0 = x_0 \longleftrightarrow x_0 = \overbrace{S \dots S}^{i+1}\,0 \right).$$

Example 2.4

The addition $+ \in \mathbb{N}^{(\mathbb{N}^2)}$ is represented by the formula:

$$x_0 = x_1 + x_2.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + i_2 = x_0 \longleftrightarrow x_0 = i_1 + i_2 \right) \tag{4.3}$$

The proof is by induction on i_2:

$\mathbf{i_2 = 0}$ because of $\boxed{4 \quad \forall x\; x+0 = x}$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 = x_0 \longleftrightarrow x_0 = i_1 + 0 \right)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + 0 = x_0 \longleftrightarrow x_0 = i_1 + 0 \right).$$

$\mathbf{i_2 = i+1}$ by ⑤ $\forall x \, \forall y \, \big(x{+}Sy = S(x{+}y)\big)$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S(i_1{+}i) = x_0 \longleftrightarrow x_0 = i_1{+}Si\right)$$

The induction hypothesis yields

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + i = x_0 \longleftrightarrow x_0 = i_1{+}i\right)$$

hence we obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(S(i_1 + i) = x_0 \longleftrightarrow x_0 = i_1{+}Si\right)$$

by the very definition of the terms involved we finally have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 + (i+1) = x_0 \longleftrightarrow x_0 = i_1{+}(i+1)\right).$$

Example 2.5

The multiplication $\cdot \in \mathbb{N}^{(\mathbb{N}^2)}$ is represented by the formula:

$$x_0 = x_1{\cdot}x_2.$$

It is enough to verify

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 \cdot i_2 = x_0 \longleftrightarrow x_0 = i_1{\cdot}i_2\right). \tag{4.4}$$

The proof is by induction on i_2:

$\mathbf{i_2 = 0}$

because of ⑥ $\forall x \; x{\cdot}0 = 0$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(0 = x_0 \longleftrightarrow x_0 = i_1{\cdot}0\right)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(i_1 \cdot 0 = x_0 \longleftrightarrow x_0 = i_1{\cdot}0\right).$$

$\mathbf{i_2 = i+1}$

by $\boxed{⑦ \quad \forall x\, \forall y\, \big(x \cdot Sy = (x \cdot y) + x\big)}$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot Si = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

which is exactly

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

The induction hypothesis yields

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot i = x_0 \longleftrightarrow x_0 = i_1 \cdot i\Big)$$

so we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

by $\boxed{⑦ \quad \forall x\, \forall y\, \big(x \cdot Sy = (x \cdot y) + x\big)}$ and (4.3) we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big((i_1 \cdot i) + i_1 = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

which is exactly

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = (i_1 \cdot i) + i_1\Big)$$

and we finally obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(i_1 \cdot (i+1) = x_0 \longleftrightarrow x_0 = i_1 \cdot (i+1)\Big).$$

Lemma 2.1

The set of representable functions is closed under composition.

Proof of Lemma 2.1:

Assume $f_1, \ldots, f_n \in \mathbb{N}^{(\mathbb{N}^p)}$ and $g \in \mathbb{N}^{(\mathbb{N}^n)}$ are represented respectively by

$$\varphi_{f_1}(x_0, x_1, \ldots, x_p), \ldots, \varphi_{f_n}(x_0, x_1, \ldots, x_p)$$

and $\varphi_g(x_0, x_1, \ldots, x_n)$. i.e., we have for all integers $i_1, \ldots, i_p, k_1, \ldots, k_n$ and $1 \leqslant j \leqslant n$:

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(f_j(i_1, \ldots, i_p) = x_0 \longleftrightarrow \varphi_{f_j}(x_0, i_1, \ldots, i_p) \Big)$$

and

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(g(k_1, \ldots, k_n) = x_0 \longleftrightarrow \varphi_g(x_0, k_1, \ldots, k_n) \Big).$$

The function

$$h = g(f_1, \ldots, f_n) \in \mathbb{N}^{(\mathbb{N}^p)}$$

defined by

$$h(i_1, \ldots, i_p) = g\big(f_1(i_1, \ldots, i_p), \ldots, f_n(i_1, \ldots, i_p)\big)$$

is represented by

$$\varphi_h(x_0, x_1, \ldots, x_p) \;=\; \exists y_1 \, \exists y_2 \, \ldots \exists y_n \, \Big(\bigwedge_{1 \leqslant j \leqslant n} \varphi_{f_j}(y_j, x_1, \ldots, x_p) \;\wedge\; \varphi_g(x_0, y_1, \ldots, y_n) \Big).$$

Indeed, by the very definition of h for every $i_1, \ldots, i_p \in \mathbb{N}$ we have

$$\vdash_c \forall x_0 \Big(h(i_1, \ldots, i_p) = x_0 \;\longleftrightarrow\; \exists y_1 \, \exists y_2 \, \ldots \exists y_n \, \Big(\textstyle\bigwedge_{1 \leqslant j \leqslant n} f_j(i_1, \ldots, i_p) = y_j \;\wedge\; g(y_1, \ldots, y_n) = x_0 \Big) \Big).$$

Therefore

$$\mathcal{R}ob. \vdash_c \forall x_0 \Big(h(i_1, \ldots, i_1) = x_0 \;\longleftrightarrow\; \exists y_1 \, \exists y_2 \, \ldots \exists y_n \, \Big(\textstyle\bigwedge_{1 \leqslant j \leqslant n} \varphi_{f_j}(y_j, i_1, \ldots, i_1) \;\wedge\; \varphi_g(x_0, y_1, \ldots, y_n) \Big) \Big).$$

□

We now turn to minimisation. We need to prove that if $A \subseteq \mathbb{N}^{n+1}$ is representable and $f \in \mathbb{N}^{(\mathbb{N}^n)}$ is some **total** function defined by minimisation the following way:

$$f(i_1, \ldots, i_n) = \mu k \;\; (k, i_1, \ldots, i_n) \in A,$$

then f is representable.

This proof requires some good amount of preliminary work.

Example 2.6

We first notice that

- for all *non-zero* integer i the following holds

$$\mathcal{R}ob. \vdash_c i \neq 0. \tag{4.5}$$

To see this, let $i = j + 1$, by the very definition of the terms involved we have

$$\vdash_c i = Sj$$

hence by $\boxed{1 \quad \forall x\ Sx \neq 0}$ we obtain

$$\mathcal{R}ob.\ \vdash_c Sj \neq 0$$

which gives the result.

- for all integers i, j such that $i \neq j$ the following holds

$$\mathcal{R}ob.\ \vdash_c i \neq j. \tag{4.6}$$

the proof is by induction on $\min\{i, j\}$:

$\min\{\mathbf{i},\mathbf{j}\} = \mathbf{0}$: this is case $\boxed{4.5 \quad \mathcal{R}ob.\ \vdash_c i \neq 0 \quad \textit{for all } i \in \mathbb{N},\quad i \neq 0}$.

$\min\{\mathbf{i},\mathbf{j}\} > \mathbf{0}$: set $k + 1 = i$ and $n + 1 = j$.

By $\boxed{3 \quad \forall x\ \forall y\ (Sx = Sy \to x = y)}$ we have

$$\mathcal{R}ob.\ \vdash_c \forall x\ \forall y\ (x \neq y \to Sx \neq Sy).$$

so that we easily obtain

$$\mathcal{R}ob.\ \vdash_c k \neq n \to Sk \neq Sn$$

which is precisely

$$\mathcal{R}ob.\ \vdash_c k \neq n \to k + 1 \neq n + 1$$

i.e.,

$$\mathcal{R}ob.\ \vdash_c k \neq n \to i \neq j.$$

By induction hypothesis we have

$$\mathcal{R}ob.\ \vdash_c k \neq n,$$

therefore by *modus ponens* we finally get

$$\mathcal{R}ob.\ \vdash_c i \neq j.$$

- The following holds

$$\mathcal{R}ob.\ \vdash_c \forall x \forall y\ \left(y \neq 0 \longrightarrow x+y \neq 0\right). \tag{4.7}$$

By ② $\forall x\ \exists y\ (x \neq 0 \to Sy = x)$ and ⑤ $\forall x\ \forall y\ \big(x+Sy = S(x+y)\big)$ and ① $\forall x\ Sx \neq 0$ we obtain

$$\mathcal{R}ob. \vdash_c \forall x \forall y \exists z\ \Big(y \neq 0 \longrightarrow (y = Sz\ \wedge\ x+Sz = S(x+z)\ \wedge\ S(x+z) \neq 0)\Big)$$

which immediately yields the result.

- The following holds

$$\mathcal{R}ob. \vdash_c \forall x \forall y\ \Big(x+y = 0 \longrightarrow (x = 0\ \wedge\ y = 0)\Big). \tag{4.8}$$

By previous result (4.7) $\mathcal{R}ob. \vdash_c \forall x \forall y\ \big(y \neq 0 \longrightarrow x+y \neq 0\big)$ we see that

$$\mathcal{R}ob. \vdash_c \forall x \forall y\ \big(x+y = 0 \longrightarrow y = 0\big).$$

and by ④ $\forall x\ x+0 = x$ we obtain immediately the result.

Notation 2.1

We introduce " $x \leqslant z$" to abbreviate the formula " $\exists y\ y+x = z$". We also introduce " $x < z$" for the formula " $\exists y\ \big(y+x = z\ \wedge\ x \neq z\big)$".

Example 2.7

We establish

$$\mathcal{R}ob. \vdash_c \forall x\ \ \neg x < 0 \tag{4.9}$$

We recall that $x < y$ stands for " $\exists z\ \big(z+x = y\ \wedge\ x \neq y\big)$".
So we need to prove

$$\mathcal{R}ob. \vdash_c \forall x\ \ \neg\exists z\ \big(z+x = 0\ \wedge\ x \neq 0\big)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x\ \ \forall z\ \big(z+x \neq 0\ \vee\ x = 0\big).$$

This is logically equivalent to

$$\mathcal{R}ob. \vdash_c \forall x\ \ \forall z\ \big(z+x = 0\ \longrightarrow\ x = 0\big).$$

Which is also logically equivalent to $\boxed{\text{4.7} \quad \mathcal{R}ob. \vdash_c \forall x \forall y \ \big(y \neq 0 \longrightarrow x+y \neq 0\big)}$.

Example 2.8

For all integers n the following holds:

$$\mathcal{R}ob. \vdash_c \forall x \ \Big[x \leqslant n \longleftrightarrow \big(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\big)\Big]. \tag{4.10}$$

The direction $(\longleftarrow)$

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(\big(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\big) \longrightarrow x \leqslant n\Big)$$

First, by making use of $\boxed{4 \quad \forall x \ x+0 = x}$ and $\boxed{5 \quad \forall x \ \forall y \ \big(x+Sy = S(x+y)\big)}$, the very definition of $k = \underbrace{S\ldots S}_{k}0$ and " $x \leqslant z$":= "$\exists y \ \big(y+x = z \ \wedge \ x \neq z\big)$", it is straightforward to establish by induction on n

$$\mathcal{R}ob. \vdash_c \forall x \ \Big[\big(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\big) \longrightarrow \exists y \ \big(y+x = n\big)\Big].$$

So it only remains to prove the direction $(\longrightarrow)$

$$\mathcal{R}ob. \vdash_c \forall x \ \Big[x \leqslant n \longrightarrow \big(x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n\big)\Big]$$

.

The proof is by induction on n:

$\mathbf{n = 0}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall x \ \big(x \leqslant 0 \longrightarrow x = 0\big)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x \ \big(\exists y \ y+x = 0 \longrightarrow x = 0\big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \big(\neg\exists y \ y+x = 0 \ \vee \ x = 0\big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \big(\forall y \ y+x \neq 0 \ \vee \ x = 0\big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \big(y+x \neq 0 \ \vee \ x = 0\big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \big(y+x = 0 \ \longrightarrow \ x = 0\big).$$

We easily obtain the result by 4.8 $\mathcal{R}ob. \vdash_c \forall x \forall y \ \ (x+y=0 \longrightarrow (x=0 \ \wedge \ y=0))$

$\mathbf{n = k+1}$: we need to show

$$\mathcal{R}ob. \vdash_c \forall x \ \ \Big(x \leqslant k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\Big).$$

which really is

$$\mathcal{R}ob. \vdash_c \forall x \ \ \Big(\exists y \ y+x=k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\Big).$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \forall y \ \ \Big(y+x=k+1 \longrightarrow (x=0 \ \vee \ x=S0 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\Big).$$

We make use of the excluded middle:

$$\vdash_c \forall y \ \ (y=0 \ \vee \ y \neq 0)$$

and distinguish between the two cases:

$\mathbf{y = 0}$**:** by 4.1 $\mathcal{R}ob. \vdash_c \forall x \ (0+x=k \ \longrightarrow \ x=k)$ we obtain

$$\mathcal{R}ob. \vdash_c \forall x \ \ (0+x=k+1 \longrightarrow x=k+1).$$

$\mathbf{y \neq 0}$**:** by 2 $\forall x \ \exists y \ (x \neq 0 \rightarrow Sy = x)$ what we need to show comes down to

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \ \Big(Sz+x=k+1 \longrightarrow (x=0 \vee x=S0 \vee \ \ldots \vee x=k \vee x=k+1)\Big).$$

by 4.2 $\mathcal{R}ob. \vdash_c \forall x \forall y \ (Sy+x=k \ \longrightarrow \ S(y+x)=k)$ we already know that

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ (Sz+x=k \ \longrightarrow \ S(z+x)=k)$$

so that we need to prove

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \ \Big(S(z+x)=k+1 \longrightarrow (x=0 \vee x=1 \vee \ \ldots \vee x=k \vee x=k+1)\Big).$$

By 3 $\forall x \ \forall y \ (Sx = Sy \rightarrow x = y)$ we only need to prove

$$\mathcal{R}ob. \vdash_c \forall x \forall z \ \ \Big(z+x=k \longrightarrow (x=0 \ \vee \ x=1 \ \vee \ \ldots \ \vee \ x=k \ \vee \ x=k+1)\Big)$$

which is

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(\exists z \ z+x = k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k \ \vee \ x = k+1\big)\Big)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(x \leqslant k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k \ \vee \ x = k+1\big)\Big).$$

By induction hypothesis one has

$$\mathcal{R}ob. \vdash_c \forall x \ \Big(x \leqslant k \longrightarrow \big(x = 0 \ \vee \ x = 1 \ \vee \ \ldots \ \vee \ x = k\big)\Big)$$

so that we obtain the result very easily.

So we have proved the following two statements:

(1) $\mathcal{R}ob. \vdash_c \forall x \forall y \ \Big(y = 0 \longrightarrow \big(x+y = k+1 \longrightarrow x = k+1\big)\Big)$ and

(2) $\mathcal{R}ob. \vdash_c \forall x \forall y \Bigg(y \neq 0 \longrightarrow \Big(x+y = k+1 \longrightarrow \big(x = 0 \vee x = 1 \vee \ldots \vee x = k\big)\Big)\Bigg).$

Therefore, by an immediate application of the excluded middle we have proved the result.

Example 2.9

For all integers n the following holds:

$$\mathcal{R}ob. \vdash_c \forall x \ \big(x \leqslant n \ \vee \ n \leqslant x\big). \tag{4.11}$$

What we need to show is

$$\mathcal{R}ob. \vdash_c \forall x \ \big(\exists y \ \ y+x = n \ \vee \ \exists y \ \ y+n = x\big).$$

We make use of the excluded middle:

$$\vdash_c \forall x \ \big(x = 0 \ \vee \ x \neq 0\big)$$

and distinguish between the two cases.

if $\mathbf{x = 0}$: we need to show

$$\mathcal{R}ob. \vdash_c \exists y \ \ y+0 = n \ \vee \ \exists y \ \ y+n = 0$$

which is immediate by $\boxed{④ \quad \forall x\ x{+}0 = x}$.

if $\mathbf{x} \neq 0$: what wee need to prove comes down to

$$\mathcal{R}ob.\ \vdash_c \forall z\ \big(\exists y\ \ y{+}Sz = n\ \vee\ \exists y\ \ y{+}n = Sz\big).$$

The proof goes by induction on n:

if $\mathbf{n = 0}$: we need to prove

$$\mathcal{R}ob.\ \vdash_c \forall z\ \big(\exists y\ \ y{+}Sz = 0\ \vee\ \exists y\ \ y{+}0 = Sz\big).$$

By $\boxed{④ \quad \forall x\ x{+}0 = x}$ this comes down to

$$\mathcal{R}ob.\ \vdash_c \forall z\ \big(\exists y\ \ y{+}Sz = 0\ \vee\ \exists y\ \ y = Sz\big)$$

which is immediate.

if $\mathbf{n = k + 1}$: we need to prove

$$\mathcal{R}ob.\ \vdash_c \forall z\ \big(\exists y\ \ y{+}Sz = k+1\ \vee\ \exists y\ \ y{+}(k+1) = Sz\big).$$

By $\boxed{⑤ \quad \forall x\ \forall y\ \big(x{+}Sy = S(x{+}y)\big)}$ and $\boxed{③ \quad \forall x\ \forall y\ (Sx = Sy \rightarrow x = y)}$ this amounts to proving

$$\mathcal{R}ob.\ \vdash_c \forall z\ \big(\exists y\ \ y{+}z = k\ \vee\ \exists y\ \ y{+}k = z\big)$$

which is exactly the induction hypothesis.

We have already proved the following results:

Lemma 2.2

(4.1) $\quad \mathcal{R}ob.\ \vdash_c \forall x\ \big(0{+}x = k\ \longrightarrow\ x = k\big)$

(4.2) $\quad \mathcal{R}ob.\ \vdash_c \forall x \forall y\ \big(Sy{+}x = k\ \longrightarrow\ S(y{+}x) = k\big)$

(4.3) $\quad \mathcal{R}ob.\ \vdash_c \forall x_0\ \big(i_1 + i_2 = x_0 \longleftrightarrow x_0 = i_1{+}i_2\big)$

(4.4) $\quad \mathcal{R}ob.\ \vdash_c \forall x_0\ \big(i_1 \cdot i_2 = x_0 \longleftrightarrow x_0 = i_1{\cdot}i_2\big)$

(4.5) $\quad \mathcal{R}ob.\ \vdash_c i \neq 0 \quad$ *(any $i \in \mathbb{N}$, $\ i \neq 0$)*

(4.6) $\quad \mathcal{R}ob.\ \vdash_c i \neq j \quad$ *(any $i, j \in \mathbb{N}$, $\ i \neq j$)*

(4.7) $\mathcal{R}ob. \vdash_c \forall x \forall y \ \big(y \neq 0 \longrightarrow x+y \neq 0\big)$

(4.8) $\mathcal{R}ob. \vdash_c \forall x \forall y \ \big(x+y = 0 \longrightarrow (x = 0 \ \wedge \ y = 0)\big)$

(4.9) $\mathcal{R}ob. \vdash_c \forall x \ \neg x < 0$

(4.10) $\mathcal{R}ob. \vdash_c \forall x \ \big(x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \dots \ \vee \ x = n)\big)$

(4.11) $\mathcal{R}ob. \vdash_c \forall x \ \big(x \leqslant n \ \vee \ n \leqslant x\big).$

Proof of Lemma 2.2:

See Examples 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9.

□

Lemma 2.3

Let $A \subseteq \mathbb{N}^{n+1}$ be representable. If the following function $f \in \mathbb{N}^{(\mathbb{N}^n)}$ is total, then f is representable.

$$f(i_1, \dots, i_n) = \mu k \ \ (k, i_1, \dots, i_n) \in A.$$

Proof of Lemma 2.3:

Assume $\varphi(x_0, x_1, \dots, x_n)$ represents the set A. We claim the function f is represented by the formula

$$\varphi(x_0, x_1, \dots, x_n) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, x_1, \dots, x_n).$$

We need to show that for all $i_1, \dots, i_n \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(i_1, \dots, i_n) = x_0 \longleftrightarrow \left(\varphi(x_0, i_1, \dots, i_n) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, i_1, \dots, i_n)\right)\right).$$

(We write $\overrightarrow{i}$ for $i_1, \dots, i_n$ and $\overrightarrow{i}$ for $i_1, \dots, i_n$)

(⇒) (1) by the very definition of f and the fact that $\varphi(x_0, x_1, \dots, x_n)$ represents A we trivially have :

$$\mathcal{R}ob. \vdash_c \varphi\left(f(\overrightarrow{i}), \overrightarrow{i}\right)$$

which is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) = x_0 \longrightarrow \varphi(x_0, \overrightarrow{i})\right).$$

(2) To show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) = x_0 \longrightarrow \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i}) \right)$$

we show the equivalent

$$\mathcal{R}ob. \vdash_c \forall y \ \left(y < f(\overrightarrow{i}) \longrightarrow \ \neg\varphi(y, \overrightarrow{i}) \right)$$

We have two cases

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{0}$: we make use of 4.9 $\mathcal{R}ob. \vdash_c \forall x \ \ \neg x < 0$ which settles this case.

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{k+1}$: by the very definition of f and the fact that $\varphi(x_0, x_1, \ldots, x_n)$ represents A we trivially have :

$$\mathcal{R}ob. \vdash_c \neg\varphi(0, \overrightarrow{i}) \ \wedge \ \neg\varphi(S0, \overrightarrow{i}) \ \wedge \ \ldots \ \wedge \ \neg\varphi(k, \overrightarrow{i})$$

which sums up to

$$\mathcal{R}ob. \vdash_c \forall y \ \left(\left(y = 0 \ \vee \ y = S0 \ \vee \ \ldots \ \vee \ y = k\right) \longrightarrow \neg\varphi(y, \overrightarrow{i}) \right)$$

by 4.10 $\mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n)\right]$ we obtain

$$\mathcal{R}ob. \vdash_c \forall y \ \left(y < k+1 \longrightarrow (y = 0 \ \vee \ y = S0 \ \vee \ \ldots \ \vee \ y = k)\right)$$

which yields

$$\mathcal{R}ob. \vdash_c \forall y \ \left(y < k+1 \longrightarrow \neg\varphi(y, \overrightarrow{i}) \right).$$

(⇐) we need to show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i}) \right) \longrightarrow f(\overrightarrow{i}) = x_0 \right).$$

We prove the result by contraposition, which means we prove

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) \neq x_0 \longrightarrow \neg\left(\varphi(x_0, \overrightarrow{i}) \ \wedge \ \forall y < x_0 \ \neg\varphi(y, \overrightarrow{i}) \right) \right).$$

By 4.11 $\mathcal{R}ob. \vdash_c \forall x \ \left(x \leqslant n \ \vee \ n \leqslant x\right)$ we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(x_0 \leqslant f(\overrightarrow{i}) \ \vee \ f(\overrightarrow{i}) \leqslant x_0\right)$$

Which is also

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \big(x_0 < f(\overrightarrow{i})\ \vee\ f(\overrightarrow{i}) < x_0\ \vee\ x_0 = f(\overrightarrow{i})\big)$$

so that we only need to prove

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \Big(\big(x_0 < f(\overrightarrow{i})\ \vee\ f(\overrightarrow{i}) < x_0\big) \longrightarrow \neg\Big(\varphi(x_0, \overrightarrow{i})\ \wedge\ \forall y < x_0\ \neg\varphi(y, \overrightarrow{i})\Big)\Big).$$

We will successively prove

(1) $\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \Big(x_0 < f(\overrightarrow{i})\ \longrightarrow\ \neg\Big(\varphi(x_0, \overrightarrow{i})\ \wedge\ \forall y < x_0\ \neg\varphi(y, \overrightarrow{i})\Big)\Big).$

We have two cases:

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{0}$: we make use of

$$4.9\quad \mathcal{R}ob.\ \vdash_c\ \forall x\ \ \neg x < 0$$

which settles this case.

if $\mathbf{f}(\overrightarrow{i}) = \mathbf{k+1}$: by

$$4.10\quad \mathcal{R}ob.\ \vdash_c\ \forall x\ \big[x \leqslant n \longleftrightarrow (x = 0\ \vee\ x = S0\ \vee\ \dots\ \vee\ x = n)\big]$$

we obtain

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \ (x_0 < k+1 \longrightarrow (x_0 = 0\ \vee\ x_0 = S0\ \vee\ \dots\ \vee\ x_0 = k))$$

and by the very definition of f:

$$\mathcal{R}ob.\ \vdash_c\ \neg\varphi(0, \overrightarrow{i})\ \wedge\ \neg\varphi(S0, \overrightarrow{i})\ \wedge\ \dots\ \wedge\ \neg\varphi(k, \overrightarrow{i})$$

which gives

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \ (x_0 < k+1 \longrightarrow \neg\varphi(x_0, \overrightarrow{i}))$$

which settles this case.

(2) $\mathcal{R}ob.\ \vdash_c\ \forall x_0\ \Big(f(\overrightarrow{i}) < x_0\ \longrightarrow\ \neg\Big(\varphi(x_0, \overrightarrow{i})\ \wedge\ \forall y < x_0\ \neg\varphi(y, \overrightarrow{i})\Big)\Big).$

By the very definition of f:

$$\mathcal{R}ob.\ \vdash_c\ \varphi\big(f(\overrightarrow{i}), \overrightarrow{i}\big)$$

Thus

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) < x_0 \longrightarrow \exists y < x_0\, \varphi(y, \overrightarrow{i}) \right)$$

i.e.,

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}) < x_0 \longrightarrow \neg\forall y < x_0\, \neg\varphi(y, \overrightarrow{i}) \right)$$

which yields what we want.

(1) and (2) finish the proof.

$\square$

Theorem 2.1: Chinese remainder Theorem

Suppose $n_0, n_1, \ldots, n_k$ are positive integers which are pairwise co-prime. Then, for any given sequence of integers $a_0, a_1, \ldots, a_k$ there exists an integer x solving the system of simultaneous congruences

$$\begin{cases} x \equiv a_0 & \bmod n_0 \\ x \equiv a_1 & \bmod n_1 \\ \quad\vdots & \\ x \equiv a_k & \bmod n_k. \end{cases}$$

Proof of Theorem 2.1:

We set

$$\alpha = \prod_{i \leqslant k} n_i$$

and notice that for each $i \leqslant k$ the two integers n_i and $\frac{\alpha}{n_i}$ are co-prime. By Bézout, there exist coefficients $c_i, d_i \in \mathbb{Z}$ such that

$$c_i \cdot n_i + d_i \cdot \frac{\alpha}{n_i} = 1$$

if we set

$$e_i = d_i \cdot \frac{\alpha}{n_i}$$

we see that

$$\begin{cases} e_i \equiv 1 & \bmod n_i \\ e_i \equiv 0 & \bmod n_j \quad \textit{(any } j \neq i\textit{)} \end{cases}$$

It follows immediately that

$$\beta = \sum_{i \leqslant k} a_i \cdot e_i$$

is a solution to the system.

□

Lemma 2.4: Gödel's β-function

There exists some function $\beta \in \mathbb{N}^{(\mathbb{N}^3)}$ which is both representable and $\mathcal{P}rim.\ \mathcal{R}ec.$ such that for all $k \in \mathbb{N}$ and every sequence $n_0, n_1, \ldots, n_k$ there exist $a, b \in \mathbb{N}$ such that

$$\begin{cases} \beta(0,a,b) & = & n_0 \\ \beta(1,a,b) & = & n_1 \\ & \vdots & \\ \beta(k,a,b) & = & n_k. \end{cases}$$

Proof of Lemma 2.4:

The function is defined[a] by

$$\beta(i,a,b) = b \dot{-} \left(\left[\frac{b}{a(i+1)+1} \right] \cdot (a(i+1)+1) \right)$$

This shows that it is $\mathcal{P}rim.\ \mathcal{R}ec.$. To show that β is representable we consider the formula

$$x_0 < S(x_2 \cdot Sx_1) \ \wedge\ \exists y \leqslant x_3 \quad \Big(y \cdot S(x_2 \cdot Sx_1)\Big) + x_0 = x_3$$

To show that this formula represents the function β, we need to show that for all $i_1, i_2, i_3 \in \mathbb{N}$

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\beta(i_1, i_2, i_3) = x_0 \longleftrightarrow x_0 < S(i_2 \cdot Si_1) \ \wedge\ \exists y \leqslant i_3 \quad (y \cdot S(i_2 \cdot Si_1)) + x_0 = i_3 \right)$$

which is left as a tedious but straightforward exercise.

Now given $n_0, n_1, \ldots, n_k$, in order to find a and b, we consider any integer m that satisfies both

(1) $m \geqslant k+1$ (2) $m! \geqslant \max\{n_0, n_1, \ldots, n_k\}$.

We set $a = m!$ so that we make sure that $a+1, a \cdot 2+1, \ldots, a \cdot k+1, a \cdot (k+1)+1$ are co-prime. To see this, we proceed by contradiction and consider there exists some prime number p that divides both $a(i+1)+1$ and $a(j+1)+1$ for some $0 \leqslant i < j \leqslant k$. Then p also divides

$$a(j+1)+1 \; - \; \big(a(i+1)+1\big) = a(j-i) = m!(j-i)$$

Since $m > (j-i)$ holds, p divides $m!$ which contradicts p divides $m!(i+1)+1$.

The Chinese Remainder Theorem (2.1) guarantees that there exists some integer b that satisfies

$$\begin{cases} b \equiv n_0 & \mod a+1 \\ b \equiv n_1 & \mod a \cdot 2+1 \\ \quad \vdots & \\ b \equiv n_k & \mod a \cdot (k+1)+1. \end{cases}$$

We chose m such that $a = m! \geqslant \max\{n_0, n_1, \ldots, n_k\}$ in order to insure $n_i < a(i+1)+1$ for every integer $i \leqslant k$. This makes certain that for each $i \leqslant k$ we have $\beta(i,a,b) = n_i$.

□

[a] $\beta(i,a,b)$ is the remainder of the division of b by $a(i+1)+1$.

Lemma 2.5

If both functions $g \in \mathbb{N}^{(\mathbb{N}^p)}$ and $h \in \mathbb{N}^{(\mathbb{N}^{p+2})}$ are representable, then the function $f \in \mathbb{N}^{(\mathbb{N}^{p+1})}$ defined by recursion below is also representable.

$$\begin{cases} f(\vec{x}, 0) & = \; g(\vec{x}) \\ f(\vec{x}, y+1) & = \; h\big(\vec{x}, y, f(\vec{x}, y)\big). \end{cases}$$

Proof of Lemma 2.5:

We let $\vec{x}$ stand for $x_1, \ldots, x_p$ and assume $g \in \mathbb{N}^{(\mathbb{N}^p)}$ and $h \in \mathbb{N}^{(\mathbb{N}^{p+2})}$ are represented respectively by $\varphi_g(x_0, \vec{x})$ and $\varphi_h(x_0, \vec{x}, x_{p+1}, x_{p+2})$.
We also consider the following formula that represents the β-function [a] :

$$\varphi(x_0, x_1, x_2, x_3) := \; x_0 < S(x_2 \cdot S x_1) \; \wedge \; \exists y \leqslant x_3 \quad \Big(y \cdot S(x_2 \cdot S x_1)\Big) + x_0 = x_3$$

Instead of $\varphi(x_0, x_1, x_2, x_3)$ we prefer the formula $\varphi_\beta(x_0, x_1, x_2, x_3)$ below which also obviously represents β but *in a strong way*.

$$\varphi_\beta(x_0, x_1, x_2, x_3) := \; \varphi(x_0, x_1, x_2, x_3) \; \wedge \; \forall y < x_0 \; \neg\varphi(y, x_1, x_2, x_3)$$

because for any integers i, k we have

$$\mathcal{R}ob. \ \vdash_c \ \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ \varphi_\beta(x_0, i, \tilde{a}, \tilde{b})\right] \longrightarrow x_0 = k\right].$$

This holds because

$$\mathcal{R}ob. \ \vdash_c \ \forall x_0 \left[x_0 \neq k \longrightarrow \neg(x_0 \leqslant k \vee k \leqslant x_0) \ \vee \ x_0 < k \ \vee \ k < x_0\right]$$

and by 4.11 $\mathcal{R}ob. \ \vdash_c \ \forall x \ \left(x \leqslant n \ \vee \ n \leqslant x\right)$, this comes down to

$$\mathcal{R}ob. \ \vdash_c \ \forall x_0 \left[x_0 \neq k \longrightarrow x_0 < k \ \vee \ k < x_0\right]$$

and by the definition of both φ_β and k we have

(1) $\mathcal{R}ob. \ \vdash_c \ \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ x_0 < k\right] \longrightarrow \neg\varphi_\beta(x_0, i, \tilde{a}, \forall\tilde{b})\right]$

(2) $\mathcal{R}ob. \ \vdash_c \ \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[\varphi_\beta(k, i, \tilde{a}, \tilde{b}) \ \wedge \ k < x_0\right] \longrightarrow \neg\varphi_\beta(x_0, i, \tilde{a}, \forall\tilde{b})\right]$.

which yields the following which is also logically equivalent to what we want:

$$\mathcal{R}ob. \ \vdash_c \ \forall\tilde{a}\forall\tilde{b}\forall x_0 \left[\left[x_0 \neq k \ \wedge \ \varphi_\beta(k, i, \tilde{a}, \tilde{b})\right] \ \longrightarrow \ \neg\varphi_\beta(x_0, i, \tilde{a}, \tilde{b})\right].$$

The formula the we choose to represent f is the following formula $\varphi_f(x_0, x_1, \ldots, x_{p+1})$. We use the notation $\overrightarrow{x} = x_1, \ldots, x_p$ so that $\varphi_f(x_0, x_1, \ldots, x_{p+1}) = \varphi_f(x_0, \overrightarrow{x}, x_{p+1}) :=$

$$\exists\tilde{a}\exists\tilde{b}\forall\tilde{i} \leqslant x_{p+1} \ \exists y \ \exists z \quad \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{x}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{x}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, x_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix}.$$

In order to show that this formula $\varphi_f(x_0, \overrightarrow{x}, x_{p+1})$ represents f, we need to prove that for all integers $i_1, \ldots, i_p, i_{p+1}$ (we write $\overrightarrow{i}$ for $i_1, \ldots, i_p$) we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1} \, \exists y \, \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \longleftrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

(⇐) We first prove

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1} \, \exists y \, \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \right).$$

We consider the sequence of integers $f(\overrightarrow{i}, 0), \ldots, f(\overrightarrow{i}, i_{p+1})$. Following the proof of Lemma 2.4, we obtain two integers a and b to make the β-function work. Since the formulas $\varphi_\beta, \varphi_g, \varphi_h$ respectively represent the functions β, g, h, we have

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(g(\overrightarrow{i}), 0, a, b\big) \ \wedge \ \varphi_g\big(g(\overrightarrow{i}), \overrightarrow{i}\big)$$

together with

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(f(\overrightarrow{i}, i_{p+1}), i_{p+1}, a, b\big)$$

and for each integer $n < i_{p+1}$

$$\mathcal{R}ob. \vdash_c \varphi_\beta\big(f(\overrightarrow{i}, n), n, a, b\big) \wedge \varphi_h\Big(f(\overrightarrow{i}, n+1), \overrightarrow{i}, n, f(\overrightarrow{i}, n)\Big) \wedge \varphi_\beta\big(f(\overrightarrow{i}, n+1), Sn, a, b\big).$$

hence we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \bigwedge_{k \leqslant i_{p+1}} \left(\begin{array}{c} \text{k} = 0 \longrightarrow \varphi_g(g(\overrightarrow{i}), \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(f(\overrightarrow{i}, k), k, a, b) \\ \bigwedge \\ \varphi_h\big(f(\overrightarrow{i}, k+1), \overrightarrow{i}, k, f(\overrightarrow{i}, k)\big) \\ \bigwedge \\ \varphi_\beta(f(\overrightarrow{i}, k+1), Sk, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{array} \right) \right).$$

from which we logically derive

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \bigwedge_{k \leqslant i_{p+1}} \exists y \, \exists z \begin{pmatrix} \mathrm{k} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, k, a, b) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, k, y) \\ \bigwedge \\ \varphi_\beta(z, Sk, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{pmatrix} \right).$$

Furthermore, we know from

$$\boxed{4.10 \quad \mathcal{R}ob. \vdash_c \forall x \ \left[x \leqslant n \longleftrightarrow (x = 0 \ \vee \ x = S0 \ \vee \ \ldots \ \vee \ x = n) \right]}$$

that

$$\mathcal{R}ob. \vdash_c \forall \tilde{i} \ \left(\tilde{i} \leqslant i_{p+1} \longleftrightarrow \left[\ \tilde{i} = 0 \ \vee \ \tilde{i} = 1 \ \vee \ \ldots \ \vee \ \tilde{i} = i_{p+1} \right] \right).$$

Thus we obtain

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \forall \tilde{i} \leqslant i_{p+1} \, \exists y \, \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, a, b) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, a, b) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, a, b) \end{pmatrix} \right)$$

and finally

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(f(\overrightarrow{i}, i_{p+1}) = x_0 \longrightarrow \exists \tilde{a} \, \exists \tilde{b} \, \forall \tilde{i} \leqslant i_{p+1} \, \exists y \, \exists z \begin{pmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{pmatrix} \right)$$

which completes the first part of the proof.

(⇒) We need to show

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \leqslant i_{p+1}\ \exists y\ \exists z \begin{bmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

which really is the following formula

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \forall \tilde{i} \left[\tilde{i} \leqslant i_{p+1} \longrightarrow \exists y\ \exists z \begin{bmatrix} \tilde{i} = 0 \longrightarrow \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y, \tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

By 4.10 $\mathcal{R}ob. \vdash_c \forall x \left[x \leqslant n \longleftrightarrow (x = 0 \vee x = S0 \vee \ldots \vee x = n)\right]$ this is equivalent to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists\tilde{a}\exists\tilde{b}\forall\tilde{i} \left[\left(\bigvee_{k\leqslant i_{p+1}} \tilde{i} = k \right) \to \exists y \exists z \begin{bmatrix} \tilde{i} = 0 \to \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y,\tilde{i},\tilde{a},\tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \to f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

which is equivalent to [b]

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists\tilde{a}\exists\tilde{b}\forall\tilde{i} \left[\bigwedge_{k\leqslant i_{p+1}} \left(\tilde{i} = k \to \exists y \exists z \begin{bmatrix} \tilde{i} = 0 \to \varphi_g(y, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y,\tilde{i},\tilde{a},\tilde{b}) \\ \bigwedge \\ \varphi_h(z, \overrightarrow{i}, \tilde{i}, y) \\ \bigwedge \\ \varphi_\beta(z, S\tilde{i}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right) \right] \to f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

which again is equivalent to

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\left(\exists\tilde{a}\exists\tilde{b}\left[\bigwedge_{k\leqslant i_{p+1}}\exists y\ \exists z\left[\begin{array}{c}\mathrm{k}=0\longrightarrow\varphi_g(y,\overrightarrow{i})\\ \bigwedge\\ \varphi_\beta(y,k,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_h(z,\overrightarrow{i},k,y)\\ \bigwedge\\ \varphi_\beta(z,Sk,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_\beta(x_0,i_{p+1},\tilde{a},\tilde{b})\end{array}\right]\right]\longrightarrow f(\overrightarrow{i},i_{p+1})=x_0\right)$$

which is equivalent to

$$\mathcal{R}ob.\ \vdash_c\ \forall x_0\left(\exists\tilde{a}\exists\tilde{b}\underbrace{\ldots\exists y_k\exists z_k\ldots}_{k\leqslant i_{p+1}}\left[\bigwedge_{k\leqslant i_{p+1}}\left[\begin{array}{c}\mathrm{k}=0\rightarrow\varphi_g(y_k,\overrightarrow{i})\\ \bigwedge\\ \varphi_\beta(y_k,k,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_h(z_k,\overrightarrow{i},k,y_k)\\ \bigwedge\\ \varphi_\beta(z_k,Sk,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_\beta(x_0,i_{p+1},\tilde{a},\tilde{b})\end{array}\right]\right]\rightarrow f(\overrightarrow{i},i_{p+1})=x_0\right)$$

and also to

$$\mathcal{R}ob. \vdash_c \forall x_0 \left(\exists \tilde{a} \exists \tilde{b} \underbrace{\ldots \exists y_k \exists z_k \ldots}_{k \leqslant i_{p+1}} \left[\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right] \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right)$$

and also to

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \underbrace{\ldots \forall y_k \forall z_k \ldots}_{k \leqslant i_{p+1}} \left(\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow f(\overrightarrow{i}, i_{p+1}) = x_0 \right).$$

Finally, making use of the following three facts:

(1) φ_β represents β *in a strong way* since we also have for all integers k, n

$$\mathcal{R}ob. \vdash_c \forall \tilde{a} \forall \tilde{b} \forall x_0 \Big[\big[\varphi_\beta(n, k, \tilde{a}, \tilde{b}) \ \wedge \ \varphi_\beta(x_0, k, \tilde{a}, \tilde{b}) \big] \longrightarrow x_0 = n \Big]$$

(2) φ_g represents g

(3) φ_h represents h

At last, by induction on i_{p+1} we show

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\underbrace{\ldots\forall y_k\forall z_k\ldots}_{k\leqslant i_{p+1}}\left(\bigwedge_{k\leqslant i_{p+1}}\begin{bmatrix}\varphi_g(y_0,\overrightarrow{i})\\ \bigwedge\\ \varphi_\beta(y_k,k,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_h(z_k,\overrightarrow{i},k,y_k)\\ \bigwedge\\ \varphi_\beta(z_k,Sk,\tilde{a},\tilde{b})\end{bmatrix}\rightarrow\bigwedge_{k\leqslant i_{p+1}}\begin{bmatrix}y_0=g(\overrightarrow{i})\\ \bigwedge\\ y_k=f(\overrightarrow{i},k)\\ \bigwedge\\ z_k=f(\overrightarrow{i},k+1)\end{bmatrix}\right).$$

$\mathbf{i_{p+1}=0}$**:** we only need to prove

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall y_0\forall z_0\left(\begin{bmatrix}\varphi_g(y_0,\overrightarrow{i})\\ \bigwedge\\ \varphi_\beta(y_0,0,\tilde{a},\tilde{b})\\ \bigwedge\\ \varphi_h(z_0,\overrightarrow{i},0,y_0)\\ \bigwedge\\ \varphi_\beta(z_0,S0,\tilde{a},\tilde{b})\end{bmatrix}\longrightarrow\begin{bmatrix}y_0=g(\overrightarrow{i})\\ \bigwedge\\ y_0=f(\overrightarrow{i},0)\\ \bigwedge\\ z_0=f(\overrightarrow{i},1)\end{bmatrix}\right)$$

which directly follows from the fact that φ_g and φ_h represent respectively g and h.

$\mathbf{i_{p+1}=n+1}$**:** we assume

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\underbrace{\ldots\forall y_k\forall z_k\ldots}_{k\leqslant n}\left(\bigwedge_{k\leqslant n}\begin{bmatrix}\varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b})\end{bmatrix} \longrightarrow \bigwedge_{k\leqslant n}\begin{bmatrix}y_0 = g(\overrightarrow{i}) \\ \bigwedge \\ y_k = f(\overrightarrow{i}, k) \\ \bigwedge \\ z_k = f(\overrightarrow{i}, k+1)\end{bmatrix}\right).$$

We only need to show

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall z_n\forall y_{n+1}\forall z_{n+1}\left(\begin{bmatrix}z_n = f(\overrightarrow{i}, n+1) \\ \bigwedge \\ \varphi_\beta(z_n, n+1, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(y_{n+1}, n+1, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_{n+1}, \overrightarrow{i}, n+1, y_{n+1})\end{bmatrix} \longrightarrow \begin{bmatrix}y_{n+1} = f(\overrightarrow{i}, n+1) \\ \bigwedge \\ z_{n+1} = f(\overrightarrow{i}, n+2)\end{bmatrix}\right).$$

which holds because:

(1) since φ_β represents β *in a strong way* we have

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall y_{n+1}\Big[\big[\varphi_\beta(f(\overrightarrow{i}, n+1), n+1, \tilde{a}, \tilde{b}) \wedge \varphi_\beta(y_{n+1}, n+1, \tilde{a}, \tilde{b})\big] \rightarrow y_{n+1} = f(\overrightarrow{i}, n+1)\Big]$$

(2) since φ_h represents h we have

$$\mathcal{R}ob. \vdash_c \forall\tilde{a}\forall\tilde{b}\forall y_{n+1}\forall z_{n+1}\Big[\big[\varphi_h(z_{n+1}, \overrightarrow{i}, n+1, y_{n+1}) \wedge y_{n+1} = f(\overrightarrow{i}, n+1)\big] \rightarrow z_{n+1} = f(\overrightarrow{i}, n+2)\Big]$$

To sum up things, we have obtained

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \underbrace{\ldots \forall y_k \forall z_k \ldots}_{k \leqslant i_{p+1}} \left(\bigwedge_{k \leqslant i_{p+1}} \begin{bmatrix} \varphi_g(y_0, \overrightarrow{i}) \\ \bigwedge \\ \varphi_\beta(y_k, k, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_h(z_k, \overrightarrow{i}, k, y_k) \\ \bigwedge \\ \varphi_\beta(z_k, Sk, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \longrightarrow \begin{bmatrix} y_{i_{p+1}} = f(\overrightarrow{i}, i_{p+1}) \\ \bigwedge \\ \varphi_\beta(y_{i_{p+1}}, i_{p+1}, \tilde{a}, \tilde{b}) \\ \bigwedge \\ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \end{bmatrix} \right).$$

Once again, since φ_β *strongly* represents β we have

$$\mathcal{R}ob. \vdash_c \forall x_0 \forall \tilde{a} \forall \tilde{b} \left[\left[\varphi_\beta(f(\overrightarrow{i}, i_{p+1}), i_{p+1}, \tilde{a}, \tilde{b}) \ \wedge \ \varphi_\beta(x_0, i_{p+1}, \tilde{a}, \tilde{b}) \right] \longrightarrow x_0 = f(\overrightarrow{i}, n+1) \right]$$

which finishes the proof.

□

[a] see the proof of Lemma 2.4.
[b] we have $(A \vee B) \longrightarrow C \equiv \neg(A \vee B) \vee C \equiv (\neg A \wedge \neg B) \vee C \equiv (\neg A \vee C) \wedge (\neg B \vee C) \equiv (A \longrightarrow C) \wedge (B \longrightarrow C)$.

Theorem 2.2

All total recursive functions are representable.

Proof of Theorem 2.2:

An immediate consequence of Examples 2.1, 2.2 and 2.3 and Lemmas 2.1 , 2.3 and 2.5.

□